PROBLEMS RELATED TO k-SCHUR FUNCTIONS AND THE REPRESENTATION THEORY OF KOSTKA POLYNOMIALS

This document is essentially a transcript of remarks from the second open problem session of the Kostka workshop, which was moderated by Tom Roby. The transcriber (Nick Loehr) takes full responsibility for any errors and garbles appearing below.

(1) Find a crystal structure on ribbon tableaux compatible with the spin statistic. This has been solved for type A domino tableaux.
(2) Prove that the four definitions of k-Schur functions (see Morse's lecture notes) are indeed equivalent.
(3) Prove that the k-atoms (see first definition of k-Schur functions in Morse's lecture notes) form a basis for the linear span of $\{J_\mu : \mu_1 \leq k\}$. Prove also that the expansion coefficients of the J_μ's in terms of the k-atoms refine the q,t-Kostka numbers.
(4) Prove that the k-q,t-Kostka polynomials lie in $\mathbb{N}[q,t]$.
(5) Prove that the (dual) k-Schur functions are the symmetric component of affine Schubert polynomials. Prove also that the k-Schur functions are a Schubert basis of the homology of the affine Grassmannian, and that the dual k-Schur functions give a Schubert basis of the cohomology of the affine Grassmannian.
(6) Generalize k-Schur functions to root systems other than type A.
(7) Elucidate the representation-theoretical significance of the k-Schur functions. The answer may involve either representation theory of symmetric groups or representation theory of Lie algebras.
(8) Describe the expansions of LLT polynomials in terms of k-Schur functions. This problem may serve as a stepping-stone towards the problem of computing Schur expansions of LLT polynomials. For instance, LLT polynomials *are* k-Schur functions in certain special cases.
(9) Note that the Schur basis of Λ is contained in the spanning set consisting of all LLT polynomials. Are there other interesting bases that have some simple relation to LLT polynomials? Lam remarked that all Hall-Littlewood polynomials are LLT polynomials, as are all skew Schur functions. Morse remarked that not every k-Schur function is an LLT polynomial (example: $s^{(3)}_{(2,1,1)}$). Modified Macdonald polynomials and (conjecturally) $\nabla(e_n)$ can be expressed as weighted sums of LLT polynomials indexed by tuples of ribbons and tuples of shifted columns, respectively.
(10) What is the relationship between k-Schur functions and k-level-restricted Schur functions? These *may* be essentially the same — it was asserted that *some* of the structure constants for k-Schur functions (the k-Littlewood-Richardson coefficients) give *all* of the structure constants for level-restricted Schur functions. A paper by Goodman

and Wenzl on Iwahori-Hecke algebras of type A at roots of unity (*J. Algebra* **215** (1999), 694—734) may be relevant in this context.

(11) Prove Buch's (weakened) version of Knutson's (false) conjecture for the Schubert structure constants that arise in Schubert calculus on flag manifolds. The original conjecture turns out to be false for full flags, but seems to be OK for two-step flags $V^a \subseteq V^b \subseteq \mathbb{C}^n$. A paper by Buch, Kresch, Tamvakis, and Yong (*Duke Math J.* **122** (2004), 125—143) reduces q-Schubert calculus on Grassmannians to this two-step case.

(12) Consider the product $B = B^{r_\ell,s_\ell} \otimes_k \cdots \otimes_k B^{r_1,s_1}$. In this case (product of rectangles in type A), there are known combinatorial interpretations and fermionic formulas for the fusion coefficients via rigged configurations (Schilling et al.). There exist conjectural formulas in other types (Hatayama et al.). The problem of generalizing the grading of the tensor product to the two-variable (q,t) case is open.

(13) Suspicions were voiced at the workshop that the two gradings of products of Schur functions — one from LLT polynomials and one from Morse and Lapointe's t-statistic — may be related. Can this relationship be stated precisely? What role does the affine Hecke algebra play? What is the connection between q-Littlewood-Richardson coefficients, Macdonald polynomials, and the definition of k-Schur functions via generalized Kostka polynomials?

(14) Are there any nice relationships between k-Schur functions and diagonal harmonics modules or Garsia-Haiman modules (whose Frobenius series are given by $\nabla(e_n)$ and $\tilde{H}_\mu$, respectively)?

(15) Define a "charge" statistic on galleries. Recall that the tableaux form a subet of the galleries (under a natural injection), so the new statistic should restrict to the usual charge statistic on tableaux. Galleries, in turn, can be viewed as Littelmann paths, so one could even ask for a charge statistic on the latter objects.

(16) Give a "non-miraculous" representation-theoretical explanation of charge. Note that there exists a well-defined parametrizing set for a basis of the representation for each weight space and filtration. Yet the canonical Lustzig-Kashiwara bases are not compatible — how can these be related?

(17) Find a good definition of level-restricted q,t-Kostka numbers. Does this make sense for root systems other than type A? Is there any connection to k-q,t-Kostka numbers?